LESSON

7 Basic Rules of Differentiation

Using the limit definition to find derivatives can be very tedious. Luckily, there are many shortcuts available. For example, if function f is a constant, like $f(x) = 5$ or $f(x) = 18$, then $f'(x) = 0$. This can be proven for all constants c at the same time in the following manner.

If:

$f(x) = c$

then:

$$f'(x) = \lim_{a\to 0}\frac{f(x+a)-f(x)}{a} = \lim_{a\to 0}\frac{c-c}{a} = \lim_{a\to 0}\frac{0}{a} = 0$$

All of the general rules in this chapter can be proven in such a manner, using the limit definition of the derivative, though we shall not bother to do so. The first rule is the Constant Rule, which says that if $f(x) = c$ where c is a constant, then $f'(x) = 0$.

Before we go any further, a word needs to be said about notation. The concept of the derivative was discovered by both Isaac Newton and Gottfried Leibniz. Newton would put a dot over an object to represent its derivative, much like the way $f'(x)$ represents the derivative of $f(x)$. Leibniz would write the derivative of y (where x is the variable) as $\frac{dy}{dx}$. Newton's notation is certainly more convenient, but Leibniz's enables us

Constant Rule

If $f(x) = c$ where c is a constant, then $f'(x) = 0$.

And, using Leibnez's notation, if c is a constant, then $\frac{d}{dx}(c) = 0$.

Power Rule

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

to represent "take the derivative of something" as $\frac{d}{dx}$ (something). Thus, if $y = f(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f(x)) = f'(x)$. Using Leibniz's notation, the Constant Rule where c is a constant is $\frac{d}{dx}(c) = 0$.

We will use both forms of the Constant Rule, depending on the situation. The next rule is the Power Rule, which is stated: $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$. This rule says "multiply the exponent in front and then subtract one from it."

Example

Differentiate $f(x) = x^2$.

Solution

$f'(x) = 2x^{2-1} = 2x^1 = 2x$

Example

Differentiate $y = x^8$.

Solution

$\frac{dy}{dx} = 8x^7$

Example

Differentiate $g(x) = \sqrt{x}$.

Solution

To use the Power Rule, we need $g(x)$ expressed as x raised to a power, or:

$g(x) = x^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

Notice how much easier it is to use the Power Rule to solve this problem than it was using the limit definition of the derivative in Lesson 6.

The Constant Coefficient Rule

If a function has a constant multiplied in front, leave it while you take the derivative of the rest.

Example

Differentiate $y = \frac{1}{x^2}$.

Solution

Again, we have to rewrite y as x^{-2} so that it becomes x raised to a power.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$= \frac{d}{dx}(x^{-2}) = -2x^{-2-1} = -2x^{-3} = \frac{-2}{x^3}$$

Example

Differentiate $y = \sqrt[3]{t}$.

Solution

$$\frac{d}{dt}(\sqrt[3]{t}) = \frac{d}{dt}(t^{\frac{1}{3}}) = \frac{1}{3}t^{\frac{1}{3}-1} = \frac{1}{3}t^{-\frac{2}{3}} = \frac{1}{3t^{\frac{2}{3}}}$$

Notice that $\frac{d}{dt}$ means "take the derivative with respect to variable t." Usually, our variable is x, so the derivative of $y = f(x)$ is $\frac{dy}{dx} = f'(x)$, but sometimes, we have other variables. If $y = f(u)$, then $\frac{dy}{du} = f'(u)$ is the derivative with respect to u, for example.

▶ Practice

Differentiate each of the following.

1. $f(x) = x^5$
2. $y = x^7$
3. $g(u) = u^{-5}$
4. $h(x) = 8$
5. $y = t^4$
6. $y = x^{\frac{7}{5}}$
7. $f(x) = x^{100}$
8. $f(t) = -11$
9. $h(x) = x$
10. $y = x^{\frac{2}{3}}$
11. $g(x) = x^{-\frac{4}{5}}$
12. $k(x) = \sqrt[4]{x}$
13. $y = \sqrt{u}$
14. $y = \frac{1}{x}$
15. $f(x) = \frac{1}{\sqrt{x}}$
16. $g(x) = \frac{1}{x^3}$

If parts of a function are added together, differentiate the parts separately.

▶ The Constant Coefficient Rule

The Constant Coefficient Rule is stated as follows: If a function has a constant multiplied in front, leave it while you take the derivative of the rest. This means that because $\frac{d}{dx}(x^8) = 8x^7$, then the derivative of $5x^8$ is $5 \cdot (8x^7) = 40x^7$. Just imagine that the constant steps aside and waits while you differentiate the rest.

Examples

Differentiate the following.

$f(x) = 11x^4$

$y = 10x^2$

$g(x) = 3\sqrt{x} = 3x^{\frac{1}{2}}$

$h(t) = \frac{4}{t^6} = 4t^{-6}$

$y = 12x$

$k(u) = \frac{15\sqrt[3]{u}}{4} = \frac{15}{4}u^{\frac{1}{3}}$

$A(r) = \pi r^2$

Solutions

$f'(x) = 44x^3$

$\frac{dy}{dx} = 20x$

$g'(x) = \frac{3}{2}x^{-\frac{1}{2}} = \frac{3}{2\sqrt{x}}$

$h'(t) = -24t^{-7} = -\frac{24}{t^7}$

$\frac{dy}{dx} = 12$

$k'(u) = \frac{5}{4}u^{-\frac{2}{3}} = \frac{5}{4u^{\frac{2}{3}}}$

$A'(r) = 2\pi r$

In that last example problem, don't forget that π is a constant, and thus $2\pi r$ should be treated just as $20r$ or $712r$ would.

Remember that $x^0 = 1$. This means that a constant function such as $f(x) = 5$ could also be written $f(x) = 5 \cdot 1 = 5x^0$. Using both the Power Rule and the Constant Coefficient Rule, it would look like this:

$f'(x) = 5(0 \cdot x^{0-1}) = 5 \cdot 0 \cdot x^{-1} = 0$

This shows that the Constant Rule really isn't necessary, because the Power Rule and the Constant Coefficient Rule together say that the derivative of a constant is zero.

▶ The Additive Rule

Next, we will examine the Additive Rule, which says that if parts of a function are added together, differentiate the parts separately. We know that

$\frac{d}{dx}(10x^2) = 20x$ and $\frac{d}{dx}(12x) = 12$. The Additive Rule then says that if $y = 10x^2 + 12x$, then $\frac{dy}{dx} = \frac{d}{dx}(10x^2 + 12x) = \frac{d}{dx}(10x^2) + \frac{d}{dx}(12x) = 20x + 12$. Because the $10x^2$ and $12x$ are added together, we differentiate them separately.

Example
Differentiate $f(x) = 4x^5 + 30x^2$.

Solution
$f'(x) = 20x^4 + 60x$

Example
Differentiate $g(x) = x^3 - 4x^2$.

Solution
This can be rewritten as addition:
$g(x) = x^3 + (-4)x^2$

thus:
$g'(x) = 3x^2 + (-4)\cdot 2x = 3x^2 - 8x$.

The previous example shows that the Additive Rule applies to cases of subtraction as well.

Examples
Differentiate the following.

$y = \sqrt{x} + 4 = x^{\frac{1}{2}} + 4$

$h(x) = 8x^5 + 10x^4 - 3x^3 + 7x^2 - 5x + 4$

$k(t) = 3t^{\frac{4}{5}} + \frac{2}{t} + 11 = 3t^{\frac{4}{5}} + 2t^{-1} + 11$

Solutions

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + 0 = \frac{1}{2\sqrt{x}}$$

$$h'(x) = 40x^4 + 40x^3 - 9x^2 + 14x - 5$$

$$k'(t) = \frac{12}{5}t^{-\frac{1}{5}} - 2t^{-2} = \frac{12}{5\sqrt[5]{t}} - \frac{2}{t^2}$$

▶ Practice

Differentiate the following.

17. $y = 3x^7$

18. $f(x) = \frac{-3}{x^{10}}$

19. $V(r) = \frac{4}{3}\pi r^3$

20. $g(t) = \frac{12t^4}{5}$

21. $k(x) = 1 - x^2$

22. $y = 4t^3 - 8t + 70$

23. $f(x) = 8x^3 + 3x^2$

24. $y = x^2 - 3x + 5$

25. $s(t) = -16t^2 + 5t + 200$

26. $F(x) = 6x^{100} + 10x^{50} - 4x^{25} + 2x^{10} - 9$

27. $g(x) = 3x^{\frac{1}{6}} + 5x^3$

28. $h(u) = u^5 + 4u^4 - 7u^3 - 2u^2 + 8u - 2$

29. $y = 3 + \frac{2}{x} + \frac{1}{x^2}$

30. $y = u^2 - u^{-2}$

31. $f(x) = 4x^2 - 8x + 5 + \frac{3}{x}$

32. $y = 4\sqrt{x} + 9\sqrt[3]{x}$

The derivative of the derivative is called the *second derivative.* The derivative of that is the *third derivative,* and so on. This is where Newton's notation really shines. If $y = f(x)$, then the derivative is $\frac{dy}{dx} = f'(x)$ and the second derivative is $\frac{d^2y}{dx^2} = f''(x)$. The third derivative is $\frac{d^3y}{dx^3} = f'''(x)$, and the tenth derivative, for example, is $\frac{d^{10}y}{dx^{10}} = f^{(10)}(x)$. We put the 10 in parentheses like that because counting the ten primes in $f''''''''''(x)$ gets ridiculous.

Example

Find the first three derivatives of $y = \sqrt{x}$.

Solution

$$y = x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$\frac{d^3y}{dx^3} = \frac{3}{8}x^{-\frac{5}{2}}$$

When working on multiple derivatives like this, it makes sense to leave the exponents negative and fractional.

Example

Find all the derivatives of $f(x) = x^3 - 4x^2 + 5x - 7$.

Solution

$$f(x) = x^3 - 4x^2 + 5x - 7$$

$$f'(x) = 3x^2 - 8x + 5$$

$$f''(x) = 6x - 8$$

$$f'''(x) = 6$$

$$f''''(x) = 0$$

All of the subsequent derivatives will also be zero, so we can write

$$f^{(n)}(x) = 0 \text{ for } n \geq 4.$$

▶ Practice

33. Find the first four derivatives of $f(x) = \frac{1}{x}$.

34. Find the second derivative of $s(t) = -16t^2 + 20t + 150$.

35. Find the third derivative of $y = 10x^4 - 7x^3 + 6x - 1$.

36. Find the first three derivatives of $y = 6\sqrt[3]{t}$.